Physics Graduate School Qualifying Examination

Spring 2019 Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** **Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
- Explain *all* variables you use in your derivations.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

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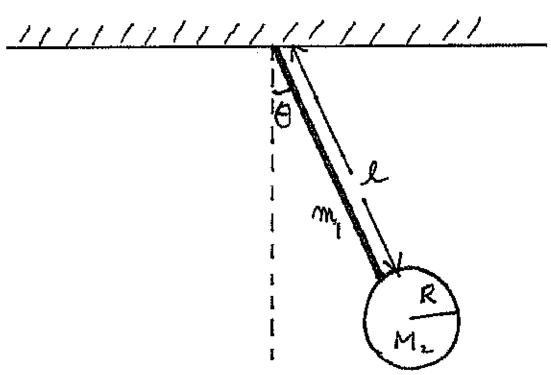
1. Consider a pendulum consisting of a thin uniform density rod of mass m_1 and length ℓ, to one end of which is fixed a sphere of mass M_2 and radius R, and about a pivot at the other end. There is no friction at the pivot. Take the gravitational acceleration to be g. Do not assume that m_1 or R are small.

(a) Compute the moment of inertia of the rod-sphere system about the pivot.

(b) Calculate the expected angular frequency of its oscillation using the small angle approximation.



2) Two identical billiard balls of radius R and mass M, rolling with velocities $\pm\vec{V}$, collide elastically head-on. Assume that after the collision they both have reversed motion and are still rolling.

a) Find the impulse that the surface of the table must exert on each ball during its reversal of motion.

b) What impulse is exerted by one ball on the other?

Recall that for a time varying force F(t) the impulse is $I = \int dt \ F(t)$

3) A particle with charge Q and mass M moves in a magnetic field, in a region of space with no currents or charges. We use cylindrical coordinates (r,ϕ,z).

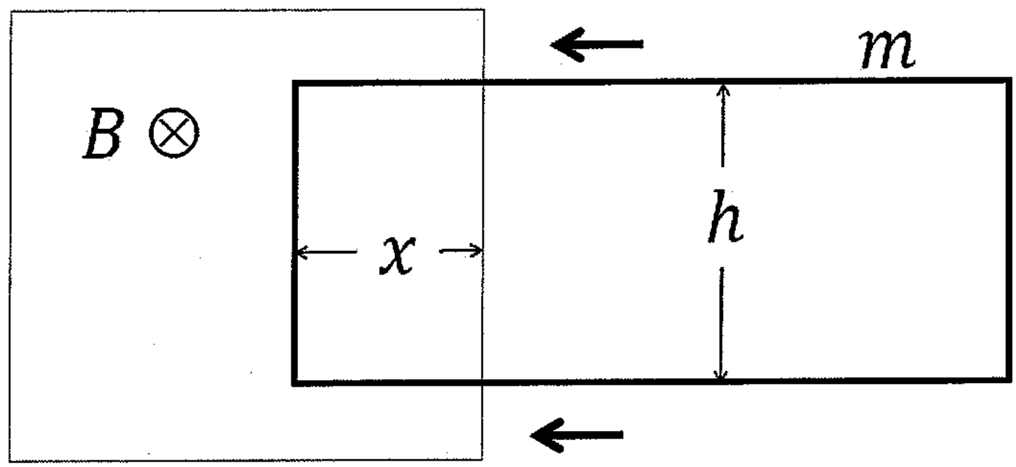
a) First, take $\vec{B}$ to be a uniform field in the z direction, $\vec{B} = B_0\hat{z}$ with $B_0 > 0$. Assume the particle moves in a circular orbit (in the xy plane) of radius R, period T, and speed v_0. Derive the cyclotron frequency $\omega_c = 2\pi / T$.

Next, suppose that B_z has a weak r dependence in the vicinity of r=R; we will examine how this may affect the unperturbed orbit discussed in (a).

b) Near r=R, $B_z(r) = B_0(1+\lambda(r-R))$. Show that the field acquires a radial component of the form $B_r = az^k$. Compute the coefficient a and the value of k.

c) The particle's motion can now acquire a z-component. Assuming that the motion r(t), ϕ(t) is unchanged, compute the motion in the *z-direction* (z(t)) and determine the condition for it to be stable or unstable.

4) Consider a rectangular wire loop of width h and mass m, part of which is in a region of magnetic field, B directed into the page. The loop has electrical resistance R and self-inductance L.



a) Derive the equations of motion for $v_x(t)$, $x(t)$, and $I(t)$, where $I(t)$ is the current in the loop.

b) If we set R=0, the loop will execute simple harmonic motion, calculate the frequency ω of that motion.

5) A quantum particle of mass m is moving in one dimension and is trapped between a rigid wall and a quadratic potential, i.e.,

$$V(x) = \frac{1}{2}kx^2 \quad \text{for } x > 0,$$

$$V(x) = \infty \quad \text{for } x < 0.$$

We are interested in the energy eigenstates of this particle.

(i) Write down the stationary Schrodinger equation, and boundary conditions, satisfied by the wavefunctions.

(ii) What subset of eigenstates of a simple harmonic oscillator satisfy the boundary conditions obtained in (i)?

(iii) Using (ii), write down the energies of the energy eigenstates of this particle.

6) Suppose that the wave function of a particle at a particular time is given by

$$\psi(r,\theta,\varphi) = \beta g(r,\theta)(e^{i\phi} + 3e^{-i\phi} - 4e^{2i\phi}).$$

Here we use spherical coordinates such that $z = r\cos(\theta), x = r\sin(\theta)\cos(\phi), y = r\sin(\theta)\sin(\phi)$. Assume that the function g satisfies the following:

$\int d^3r \mid g(r,\theta) \mid^2 = 1$, where the integral is over all space.

a) Determine the normalizing constant β.

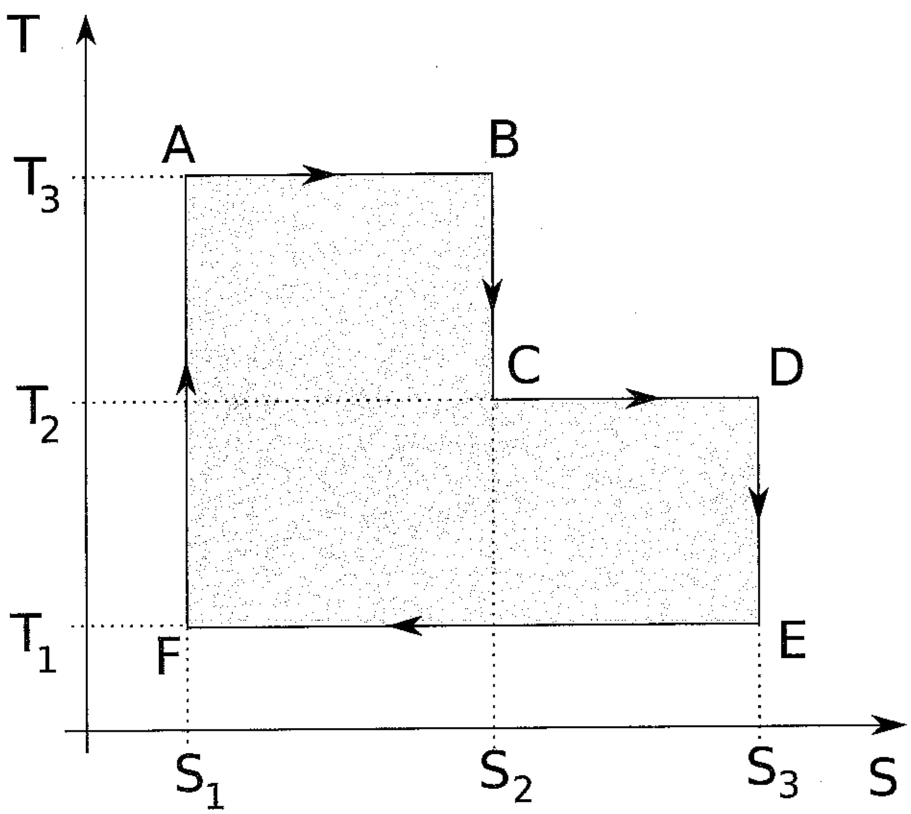
b) If we measure L_z, what are the values we could find, and what is the probability of each?

(Recall that $L_z = -i\hbar\frac{\partial}{\partial\phi}$).

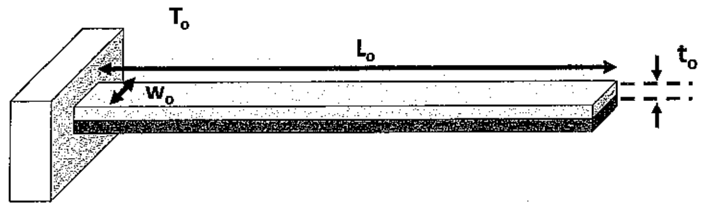
7) Consider a thermal engine that undergoes a reversible cycle as in the figure (in the S-T plane). The values $S_{1,2,3}$ and $T_{1,2,3}$ are assumed to be known and all results should be computed in terms of them. The cycle can be broken into six steps: A → B, B→C,.... Let Q_j denote the heat absorbed from a reservoir during step j; we define Q_j to be positive if the system absorbs a net heat, and negative if the system releases a net heat. Let Q_A be the sum of the Q_j over the steps where $Q_j > 0$.

a) Compute the efficiency of the engine, defined as $\eta = W/Q_A$ where W is the total work produced.

b) What is the maximum possible efficiency of a thermal engine operating between the temperatures T_3 and T_1? Show explicitly that the cycle in a) has an efficiency less than or equal to this maximum value.



8) Two thin rectangular metallic strips are made from different materials. They have the same width w_o, thickness t_o and length L_o at some temperature T_o. The two metal strips are tightly bonded together and are rigidly supported from one end as shown in the diagram. The upper metal strip has a coefficient of thermal expansion α_1 while the lower has a coefficient of thermal expansion α_2. Assume $\alpha_1 > \alpha_2$.



If the composite metal strip is uniformly heated to a temperature $T>T_o$, which way does the strip bend and what radius of curvature will it have? (Neglect the effects of gravity.) Note that $\alpha = \frac{1}{L}\frac{dL}{dT}$

Useful Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$F = -kT \ln Z \qquad Z = \sum_i e^{-\frac{E_i}{kT}} \qquad dF = -SdT - pdV + \mu dN$$

$$Z_{classical} = \frac{1}{N! h^{3N}} \int d^{3N} p \int d^{3N} r \; e^{-H/kT}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \qquad -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$$

$$\int_0^\infty dx \; x^N e^{-x} = N! \qquad \int_{-\infty}^{\infty} dx \; x^2 e^{-x^2} = \sqrt{\pi}/2 \qquad \int_{-\infty}^{\infty} dx \; e^{-x^2} = \sqrt{\pi}$$

In cylindrical coordinates r, φ, z:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r}(rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \times \vec{A} = \hat{r}\left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{\hat{z}}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)$$

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Spring 2019 Part II

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1) A sphere with radius R and mass of M is suspended on a spring in a container. The container is empty at first and then filled with a liquid of unknown viscosity η . The frequencies are measured to be ω_1 in the empty container and ω_2 in the full container. The frictional force exerted on the moving sphere is given by $F = -6\pi\eta R\text{v}$, where v is the sphere's velocity.

Assuming that the density of the fluid is much less than the density of the sphere:

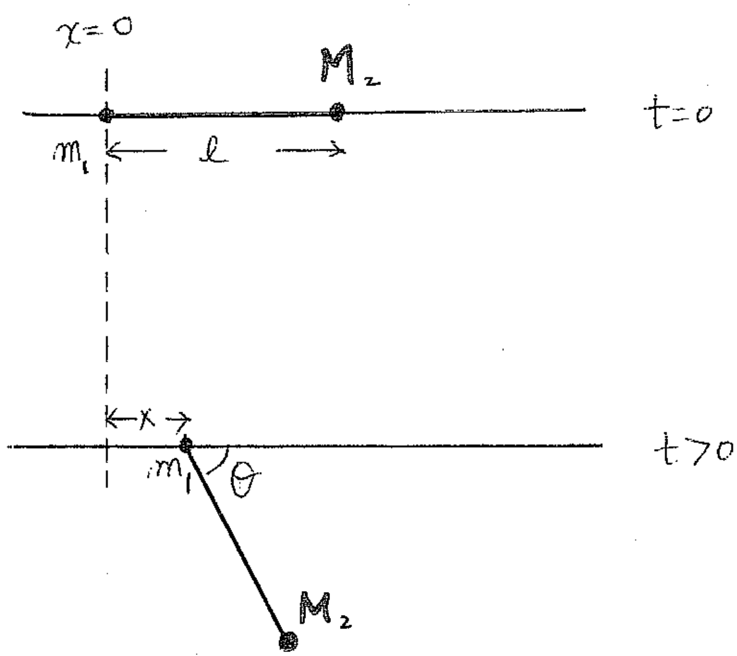
Calculate the viscosity η.

2. A point mass m_1 can slide freely on a smooth straight horizontal rod. A particle of mass M_2 is attached to it by a massless string of length ℓ. The particle is initially in contact with the rod with the string taut. It is then released and falls under gravity. The angle θ is measured from the horizontal line on which point mass m_1 moves as indicated. Take the gravitational acceleration to be g.

(a) Construct the Lagrangian for the combined system and write down the equation of motion.

(b) Write down the system energy. Is the system energy conserved?

(c) If $M_2=m_1$, what is the tension in the string when $\theta = \pi / 6$?



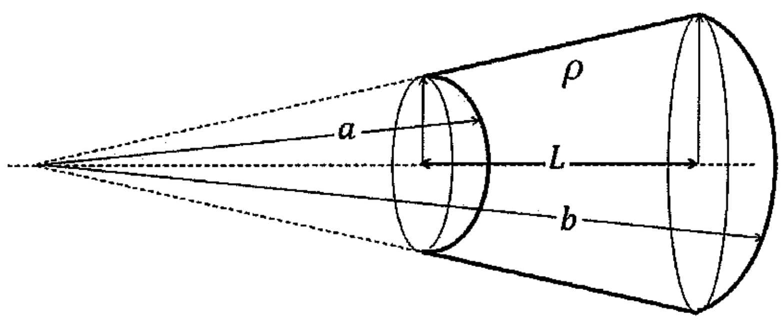
3) Consider a conducting sphere of radius R, in contact with air. Assume that the dielectric constant of the air is ε_0

a) If the sphere carries a net charge Q, what is its electrostatic energy?

b) Suppose that the sphere is connected to ground by a thin conducting wire, and that the system is in thermal equilibrium at absolute temperature T. On average the sphere has $\langle Q \rangle = 0$. Compute the mean square thermal fluctuation in charge, $\langle Q^2 \rangle$.

4) The object shown is a truncated cone made with material of resistivity ρ; the two ends are *hemispherical* surfaces of radii a and b respectively, centered at the apex of the cone, and L is the distance between the centers of the circular perimeters of the end caps as shown. Calculate the resistance R of this object.

The resistance R is defined as follows: we maintain a voltage difference V between the two hemispherical surfaces, and a current flows given by I=V/R.



5) The wave function of a particle in one dimension is given by

$$\psi(x) = Ae^{-(x-x_0)^2/b^2} e^{ikx}$$

Here, x_0, b, and k are real constants.
a) Compute the normalizing constant A.

b) Compute $< x >$.

c) Compute $< p >$.

d) Compute $\sigma_x^2 = < x^2 > - < x >^2$.

6) Consider a free particle of mass M, moving in one dimension in the positive x direction, in a quantum energy eigenstate in the lab frame. In this lab frame, the coordinate and time labels are denoted by x and t respectively, and the wavefunction, $\psi(x,t)$, has wavelength λ.

Now consider the same quantum state observed from a reference frame moving with velocity v, where the coordinates, (x', t'), are related to the lab frame coordinates by the Galilean transformation: $x' = x - vt$ and $t' = t$. The wavefunction of the particle, as observed from the moving frame, is $\tilde{\psi}(x',t')$.

(a) Using de Broglie's formula or otherwise, find the relation between the wavelength, λ', of the particle in the moving frame, and the wavelength, λ, in the lab frame.

(b) The wavefunctions $\psi(x,t)$ and $\tilde{\psi}(x',t')$ satisfy the Schrodinger equation written down in the lab and moving reference frames, respectively. Using the ansatz

$\psi(x,t) = e^{Ax+Bt}\tilde{\psi}(x - vt, t)$ where A and B are constants to be determined, find the values of A and B, using the Schrodinger equations in the two reference frames.

7) Consider a monoatomic ideal gas with N atoms in a volume V at temperature T. Note that we are assuming that the atoms do not interact with each other.

a) Compute the classical partition function.

b) Consider a given, small volume $v \ll V$. Compute the probability of a fluctuation that leaves the volume v empty (with no molecules).

8) A rigid cylinder of volume V is filled at temperature T_1 with N_1 atoms of an ideal monatomic gas until a pressure P is reached. Similarly, a cylinder of volume 2V is also filled with the same type of gas, at temperature T_1, until the same pressure P is reached. Finally, the cylinder of volume 2V is cooled from T_1 to T_2. Let S_i be the total entropy of the two cylinders at this point.

The two containers are then isolated from the rest of the universe, and placed in thermal contact with each other; after they reach equilibrium, the total entropy is S_f. What is the change in entropy of the system (the two cylinders), $S_f - S_i$?

Express your final answer in terms of the number of molecules in the smaller container (N_1) and the temperatures of each gas (T_1 and T_2). Assume the thermal mass of the two containers is negligible.

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