Physics Graduate School Qualifying Examination

Spring 2018 Part I

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- Correct answers without adequate explanation/reasoning will not receive full credit.
- Explain *all* variables you use in your derivations.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

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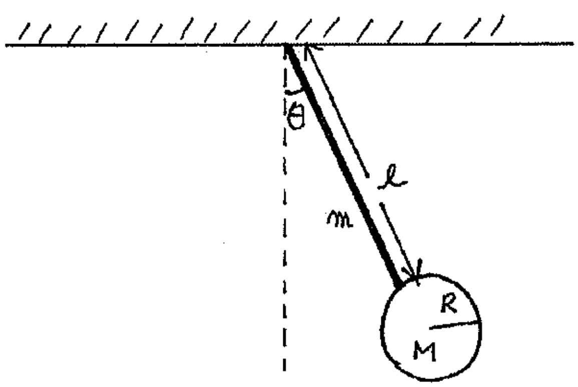
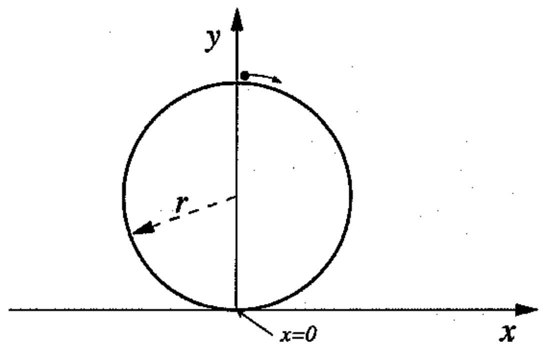
1. Consider a pendulum consisting of a thin uniform density rod of mass m_1 and length ℓ, to one end of which is fixed a solid sphere of mass M_2 and radius R, and about a pivot at the other end. There is no friction at the pivot.

   (a) Compute the moment of inertia of the rod-sphere system about the pivot. (Do not assume that R is small.)

   (b) Calculate the expected frequency of its oscillation using the small angle approximation; take the gravitational acceleration to be g.



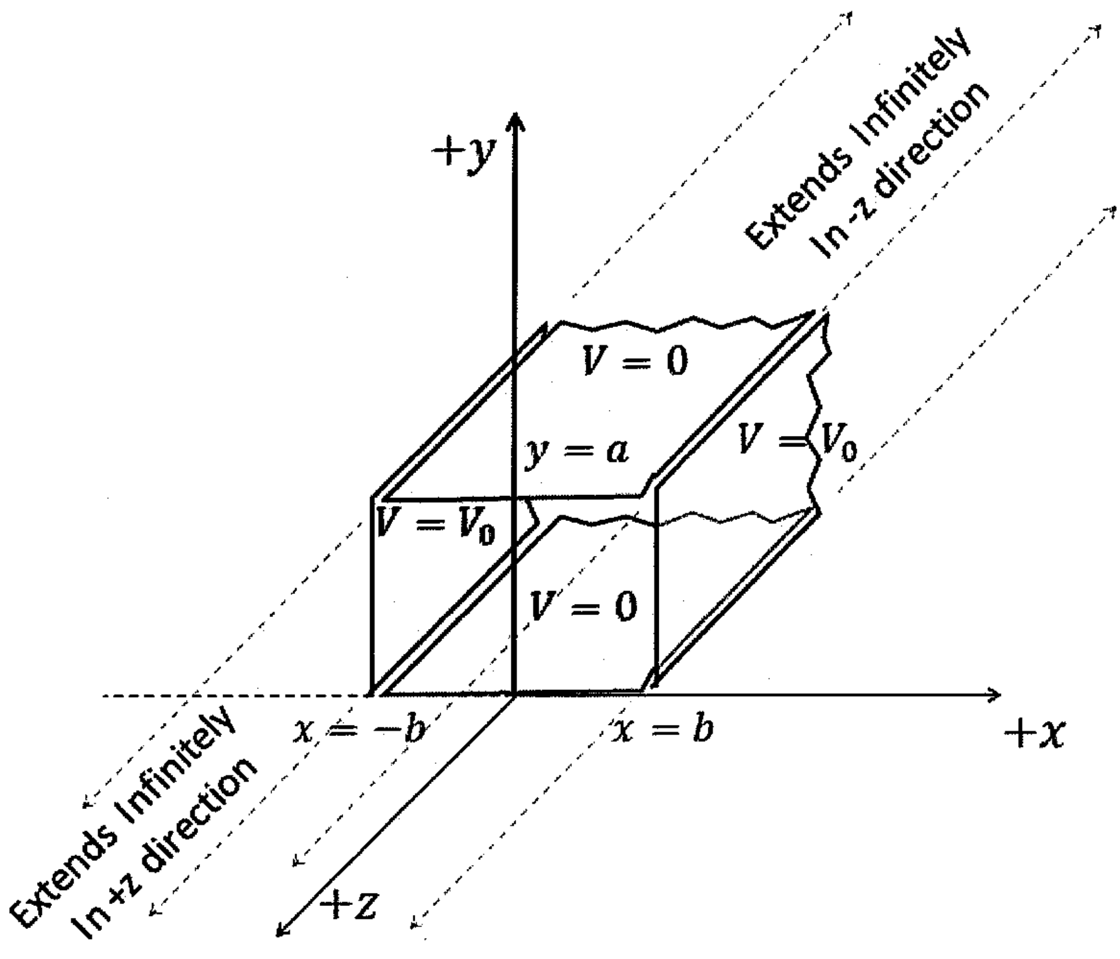
2) A point mass sits on a smooth non-moving sphere at the highest point of the sphere with radius r. Once the point mass moves to one side it will slide downwards on the surface of the sphere. Take the gravitational acceleration to be g, and note that the mass does not roll, it slides with no friction.



a) At which height h does the mass point leave the surface of the sphere?

b) At what distance x_{impact} (from x=0) does the point mass hit the horizontal ground level?

3) Two grounded metal plates at $y = 0$, and $y = a$, that extend infinitely in the +z and –z directions, are connected at $x = \pm b$ by metal strips (also extending infinitely in the +z and –z directions) maintained at a constant electrostatic potential V_0 as shown in the figure below. A thin layer of insulation at each corner prevents them from shorting out. Find the potential V(x,y) inside the resulting rectangular pipe.



4) A conducting metal solid sphere carries a total charge Q.

a. How is the charge distributed?
b. Calculate the electric force on any charge element (dq) from Q, due to the rest of the charges in Q.
c. The sphere is cut into two equal halves. How much force is needed in order to still keep the two hemispheres together? (Note that the charge distribution is unchanged.)

5) A particle in one dimension is trapped between two rigid walls,

$V(x) = 0$ for $0 < x < L$ and ∞ otherwise.

(i) Write down the energy eigenfunctions and eigenvalues of the Hamiltonian.

(ii) At $t = 0$ the particle is known to be exactly at $x = L/2$ with certainty. What are the *relative* probabilities for the particle to be found in various energy eigenstates?

(iii) Write down the wave function for $t \geq 0$, correct up to an overall normalization.

(For (ii) and (iii), you need not worry about absolute normalization, convergence, and other mathematical subtleties.)

6) Consider a quantum particle, in a one-dimensional potential given by

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ V_0 & x > a \end{cases}$$

We are looking for an energy eigenstate with $E < V_0$, i.e. a bound state.

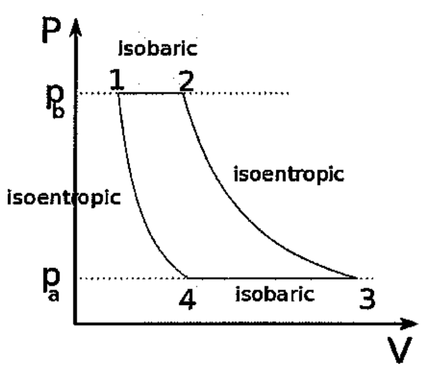
A) Write down a (unnormalized) wavefunction for the region $0<x<a$ and also for $x>a$
B) Write down the boundary conditions at $x=a$
C) Using the answer in part B), give an equation to constrain the allowed values of the energy
D) What is the constraint on a and V_0 for there to be at least one bound state?

7) Consider the reversible Brayton cycle for a gas (see PV diagram). It consists of two isobaric (constant pressure) and two isoentropic (constant entropy) paths. Assume that the substance undergoing the cycle is an ideal gas with specific heat $C_V = \alpha k$ where N is the number of molecules, k the Boltzmann constant and α a numerical constant. Also assume that the (absolute) temperatures T_1, T_2, T_3, T_4 at the points 1,2,3,4 are known.

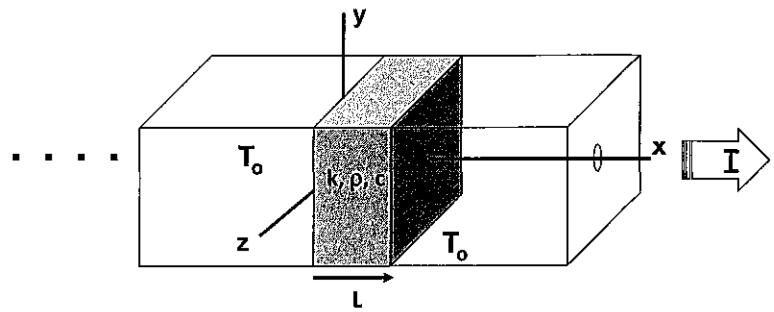
a) Compute the heat absorbed or released by the gas in each of the four steps of the cycle (Q_{12} for $1 \rightarrow 2$, Q_{23} for $2 \rightarrow 3$, Q_{34} for $3 \rightarrow 4$, Q_{41} for $4 \rightarrow 1$). Use $Q_{ij}>0$ for heat absorbed ($Q_{ij}<0$ for heat released). Express your answers in terms of the temperatures T_1, T_2, T_3, T_4.

b) Compute the total work W produced by one complete cycle. Compute the efficiency of the engine $\eta = W/Q_{in}$. Here Q_{in} is the heat absorbed by the gas, during the step when it absorbs heat (not the total heat transfer).

c) Use the isoentropic condition to derive a relation among the temperatures T_1, T_2, T_3, T_4. Use it to express the efficiency η solely in terms of T_1, T_4.



8) A large thin rigid plate of thickness L is made from a weakly conducting isotropic material. The surfaces of the plate are at x=0 and at x=L. The two surfaces of the plate are held fixed at a temperature T_o. At time t=0, an electric current passes uniformly through the plate in the x-direction and generates energy inside the plate at a rate of α W/m^3.



The equation describing the temperature profile inside the plate is

$$\frac{\partial}{\partial x}\left[k\frac{\partial T}{\partial x}\right]+\alpha=\rho c\frac{\partial T}{\partial t} \ .$$

where the thermal conductivity k, the mass density ρ, and the specific heat c are all temperature independent. <u>The plate is very wide, so that we can ignore the y and z dependence of the temperature.</u>

After a long time, what is the maximum temperature inside the plate and at what value of x does it occur?

Useful Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$\vec{m} = \frac{1}{2} \int d^3 r \ \vec{r} \times \vec{J}(\vec{r})$$

Lorentz Transformation Equations:

$$x' = \gamma(x - vt) \quad t' = \gamma(t - \frac{v}{c^2} x) \quad y' = y \quad z' = z \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E'_x = E_x \quad B'_x = B_x$$

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$$F = -kT \ln Z \qquad Z = \sum_i e^{-\frac{E_i}{kT}} \qquad dF = -SdT - pdV + \mu dN$$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \qquad -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Spring 2018 Part II

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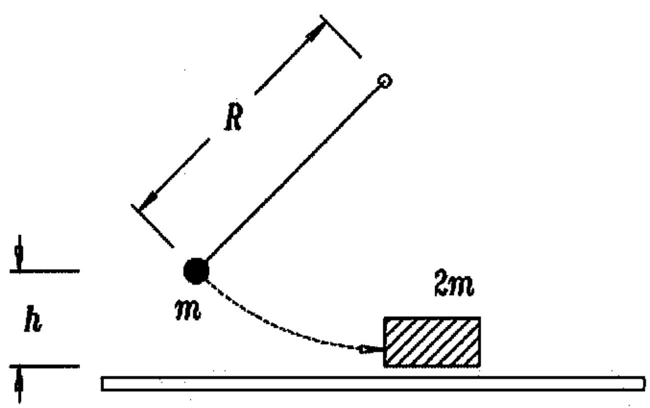
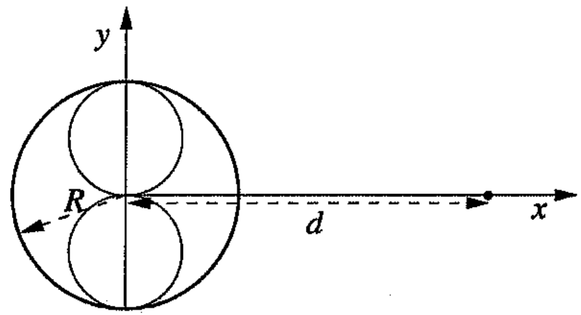
1. A pendulum consists of a massless rigid rod of length R with a weight of mass m attached to its end. The mass m is released from a height h above a table, upon which is resting a block of mass 2m, whose end is exactly below the point of suspension. There is no friction anywhere. The bob collides elastically with the block. To what height does it rise? Does it rebound or continue in the same direction? Take the gravitational acceleration to be g.



2) A lead ball with radius R has two identical spherical voids with radius R/2 (see sketch). The lead has mass density ρ. Let $M = (4/3)\pi R^3 \rho$. For all parts of this problem, a test mass m_t is on the x-axis (as shown in the figure), at a distance of d from the center of the ball.



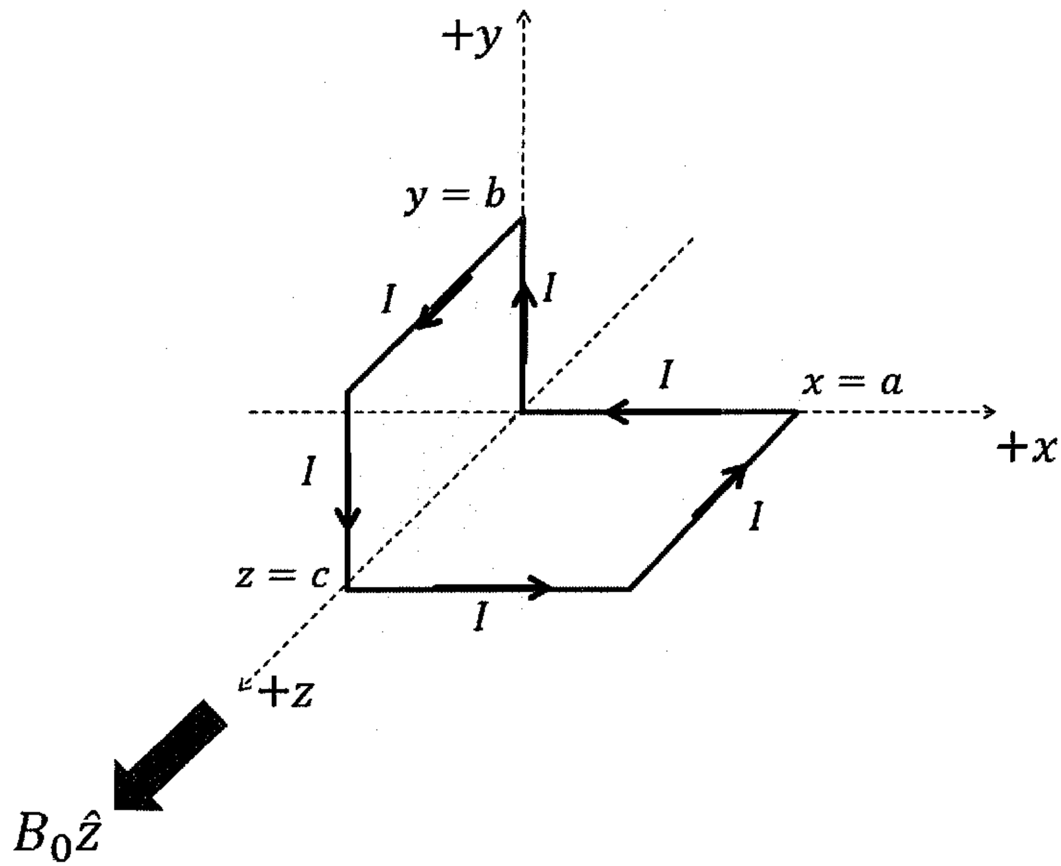
Compute the gravitational force on the test mass for:

i) d greater than R.

ii) d=R

iii) d less than R.

3) <u>Find the potential energy and the torque</u> on a current carrying magnetic loop shown in the figure below, in the presence of an external magnetic field $B_0\hat{z}$



4) An infinite length straight wire of radius r_0 carries, in the lab frame, a uniform current i, and zero net-charge density. The current is produced by a high speed electrons (speed u), and the positive ions are at rest. An observer moves with relativistic speed v along the direction of the wire.

a. What are the electric and magnetic fields in the lab frame?
b. What are the electric and magnetic fields seen by the observer?
c. What net-charge density would the observer measure?
d. What are the electron and ion speeds measured by the observer?
e. How do you explain the finite net-charge density the observer measures?

5) A spin 1/ 2 system is known to be in an eigenstate of $\vec{S} \cdot \hat{n}$ with an eigenvalue $+\hbar / 2$. $\hat{n}$ is a unit vector given by

$$\hat{n} = \hat{x}\sin(\theta) + \hat{z}\cos(\theta) \; .$$

(i) Suppose S_x is measured, what is the probability of getting $+\dfrac{\hbar}{2}$?

(ii) Evaluate the dispersion in S_x, i.e., $\sigma^2 = \; < (S_x - < S_x >)^2 > \; .$

6) Consider a spin-1/2 particle that has a spin wavefunction, $\chi = A\begin{pmatrix} 7i \\ -2 \end{pmatrix}$
The two elements represent the usual spin up (top) and spin down (bottom) elements of the z-component of the spin.

a) Normalize this spin wavefunction by finding the value of A.
b) What is the probability of measuring the spin down of the z-component of the spin?
c) Find the expectation value of $\langle S_y \rangle$.

7) Consider a non-interacting monatomic gas such that each atom has spin ½. In the presence of a constant and uniform magnetic field, the Hamiltonian of the atom is modified to (H_0 is without magnetic field)

$$H = H_0 - B\,\mu\,S_{1z}$$

where μ is the magnetic moment and we assumed the field is in the z direction.

a) Compute the partition function and from there the free energy F of the system (consider only the magnetic part, ignore the kinetic energy) .

b) Compute the magnetization $M = -\frac{\partial F}{\partial B}$ as a function of the temperature T.

c) Compute the susceptibility $\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0}$.

8) A uniform bar of radius R and length L is placed between two reservoirs having temperatures T_1 and T_2 ($T_1>T_2$). The bar has a density ρ and a specific heat c. After a long time, the bar reaches a steady state. The bar is then removed from the two reservoirs and allowed to reach thermal equilibrium <u>with no heat lost or gained to the surroundings</u>.

a) What is T(x), the equilibrium temperature distribution in the bar after it has been connected between the two reservoirs for a long time?

b) What is the final temperature of the bar after it is removed from the two reservoirs and reaches thermal equilibrium?

c) What is the entropy change of the bar?

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