Physics Graduate School Qualifying Examination

Fall 2019 Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet. Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

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- Explain *all* variables you use in your derivations.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

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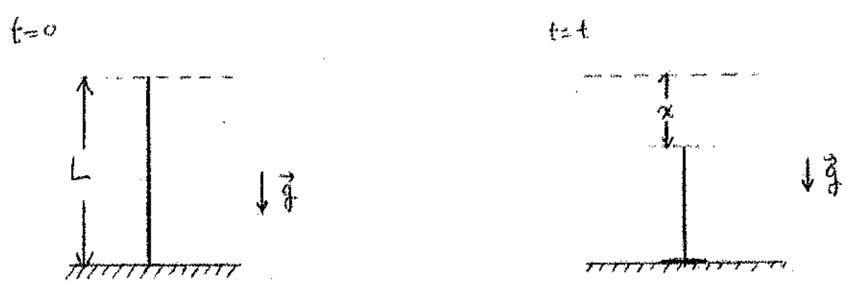
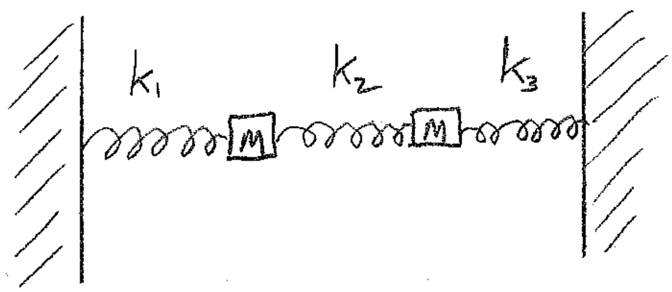
1. Consider a very thin, flexible chain of mass per unit length ρ and length L suspended just above a table. If the chain is released from rest at the top, find the force on the table when a length x of the chain has dropped to the table. Take the gravitational acceleration to be g.



2) Consider the following system: we have two masses, each of mass M, connected by three springs with spring constants k_1, k_2, and k_3. The masses execute small oscillations about their equilibrium positions. The motion is horizontal, and gravity is absent.

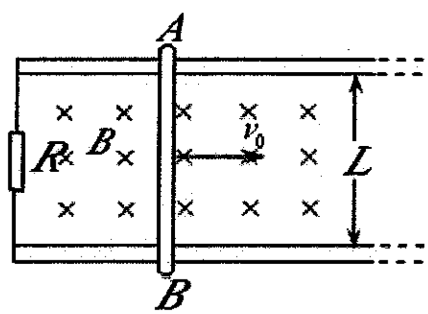
Compute the normal mode frequencies of this system.



3) A pair of parallel metal tracks are separated by a distance L, and connected by a resistor R at one end. A metal bar (mass m) slides on the tracks, without friction, to the right with initial velocity v_0. The whole system is in a constant magnetic field B perpendicular to the track plane. Ignore the resistance in the metal tracks and in the metal bar.

a) Find the distance the metal bar will slide.

b) Calculate the total heat dissipated by the resistor, and show that it is equal to the initial kinetic energy of the bar.



4) A rigid disk of radius R, thickness d, and uniform charge density ρ_0 is located with its center at the origin. The disk is rotating as shown with constant angular velocity $\omega\hat{z}$; this rotation creates a magnetic field. We may write $B_z(0,0,a)$, the z-component of the magnetic field at the point x=y=0,z=a, in the following form (assume a>d/2):

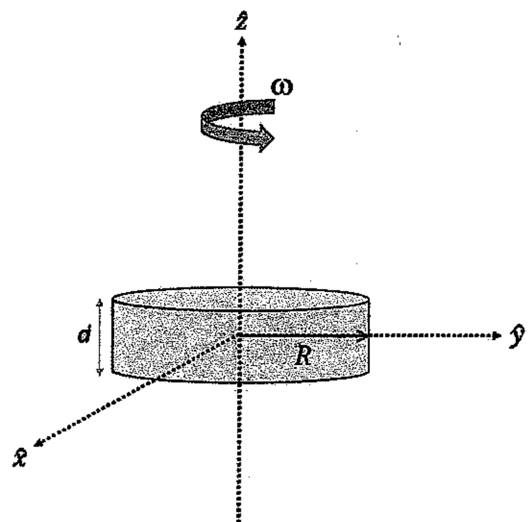
$$B_z(0,0,a) = \int_0^{2\pi} d\phi' \int_{-d/2}^{d/2} dz' \int_0^R dr' F(r',z',\phi')$$

where we use cylindrical coordinates:

$$x = r\cos(\phi), y = r\sin(\phi), z = z$$

a) Find the function $F(r,z,\phi)$

b) Find the leading term for F, when a>>d and a>>R, and evaluate the integral.



5) A quantum mechanical particle with mass m is trapped inside a one-dimensional infinite potential well of length L.

$V(x) = 0$ for $0<x<L$, $V(x) = \infty$ otherwise .

It is initially in the ground state. At t = 0, the right wall is suddenly moved away such that the well size becomes 2L.

(i) What is the energy of the particle right before the wall was moved?

(ii) List the energy eigenstates and eigenvalues for the system after the wall is moved.

(iii) At t = 1second, what are the probabilities that the particle will be found in the 2nd and 4th energy eigenstates? (Assume that eigenstates are counted in order of increasing energy, and that the first state is the ground state.)

6) A spin-1/2 particle is in the state $|\psi\rangle = \sqrt{\frac{1}{3}}\,|\vec{k}\rangle \otimes |\leftarrow\rangle + \sqrt{\frac{2}{3}}\,|-\vec{k}\rangle \otimes |\rightarrow\rangle$, where $|\vec{k}\rangle$ and $|-\vec{k}\rangle$ are two normalized eigenstates in the momentum space, $|k| \neq 0$, $|\leftarrow\rangle$ and $|\rightarrow\rangle$ are the normalized eigenstates of spin-1/2 along the x axis.

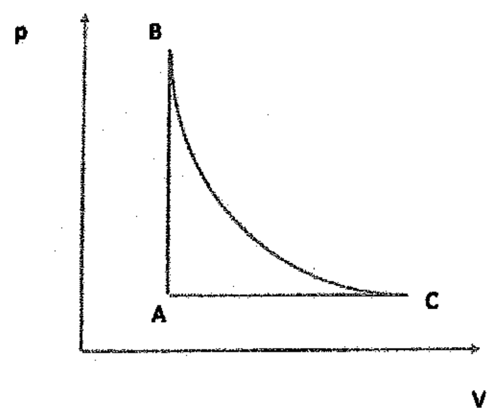
Thus, $S_x |\rightarrow\rangle = \frac{\hbar}{2} |\rightarrow\rangle \quad S_x |\leftarrow\rangle = -\frac{\hbar}{2} |\leftarrow\rangle$

Suppose we measure S_z.

a) What are the possible values we will measure?

b) What is the probability for each value?

7) An ideal monatomic gas is taken around the cycle, ABCA, in the pressure-volume diagram as shown below. The path from B to C is at constant temperature. Assume the pressure at the points are p_A, p_B, p_C, the temperature is T_A, T_B, T_C, the volume is V_A, V_B, V_C, and the gas has N particles. (Note that $T_B = T_C$, $V_A = V_B$, and $p_A = p_C$.)



A) Calculate the total work done on the gas for this full cycle
B) Calculate the change in entropy from point C to point A

8) Consider a (non-ideal) gas contained in a vessel at temperature T, pressure p and volume V that expands into a vacuum of small volume dV<<V (see figure). The whole system is thermally isolated from the environment. Work to first order in the small quantity dV.

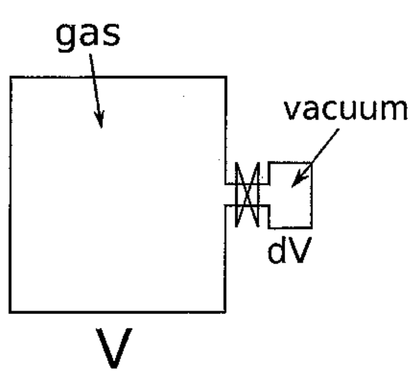
a) What is the change in energy of the gas?

b) Compute the change in entropy dS as a function of p,T and dV. Is it positive or negative? Explain why.

c) Using the result in b) and the hint below, compute the change in temperature dT in terms of p, T, dV, the specific heat $C_V = T\left(\frac{\partial S}{\partial T}\right)_V$ and $A = \left(\frac{\partial p}{\partial T}\right)_V$.

Hint: consider S(T,V) and use $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = A$

d) From c), what condition has to be satisfied by A for the gas to cool down when expanding? What happens for an ideal gas?



Useful Equations

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int d^3r'\,\frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$dU = TdS - pdV + \mu dN$$

$$F=-kT\ln Z \qquad Z=\sum_i e^{-\frac{E_i}{kT}} \qquad dF=-SdT-pdV+\mu dN$$

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi \qquad -\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi=E\psi$$

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\sum_{k=0}^{\infty}x^k=1+x+x^2+\ldots=\frac{1}{1-x}\quad\text{for } |x|<1$$

$$\int_{-\infty}^{\infty}dx\exp(-x^2)=\sqrt{\pi} \qquad \int_{-\infty}^{\infty}dx\,x^2\exp(-x^2)=\frac{\sqrt{\pi}}{2}$$

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Fall 2019 Part II

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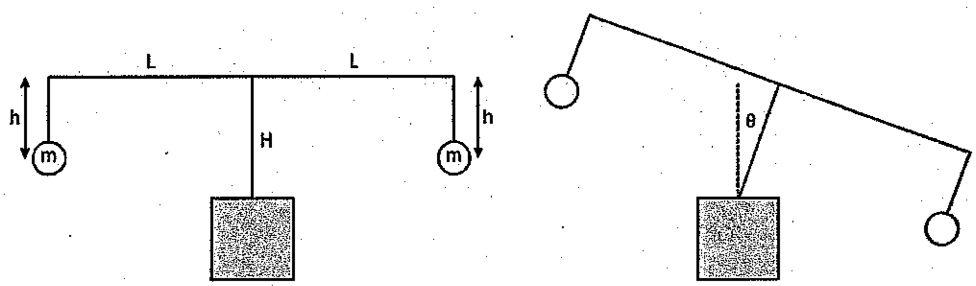
1) The figure below depicts a mass balance, wherein the rigid, symmetric, massless structure (composed of right angles) supports two equal masses and is allowed to pivot in one angular dimension as depicted at the right side.



a) Construct the potential energy for this system as a function of the angle θ, and evaluate the conditions for stability about the configuration shown at left (θ = 0). The gravitational acceleration is given by g.

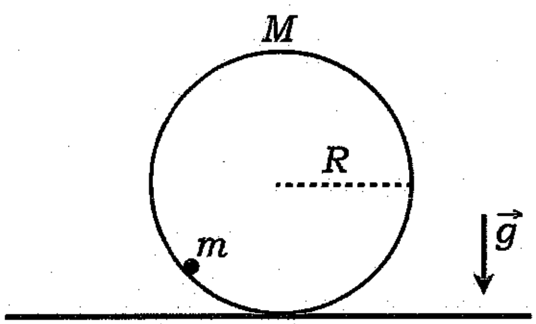
b) Calculate the oscillation frequency of this apparatus, about the point θ = 0; assume that parameter values satisfy the stability conditions from part (a). Assume that θ is small.

2. A thin circular loop of radius R, with mass M distributed uniformly along its circumference, is free to roll along a horizontal surface without slipping. A point particle of mass m is attached to the inside of the loop and is constrained to slide along the inside perimeter of the loop without friction. The system is in a uniform gravitational acceleration g.

a) Write down the Lagrangian for this system.

b) Find the equations of motion and any possible equilibrium positions for the particle.

c) Find the frequency of small amplitude oscillations of the particle about all possible positions of stable equilibrium. Consider your results in the limit $M \gg m$, and discuss.



3) A point charge q is outside a neutral, solid conducting sphere of radius R, a distance r away from the sphere center. For parts (a), (b), (c) and (d) we assume the sphere is not grounded.

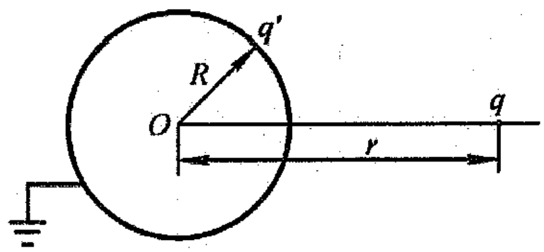
a) Find the electric field at the center of the sphere.

b) What is net total charge on the surface of the sphere?

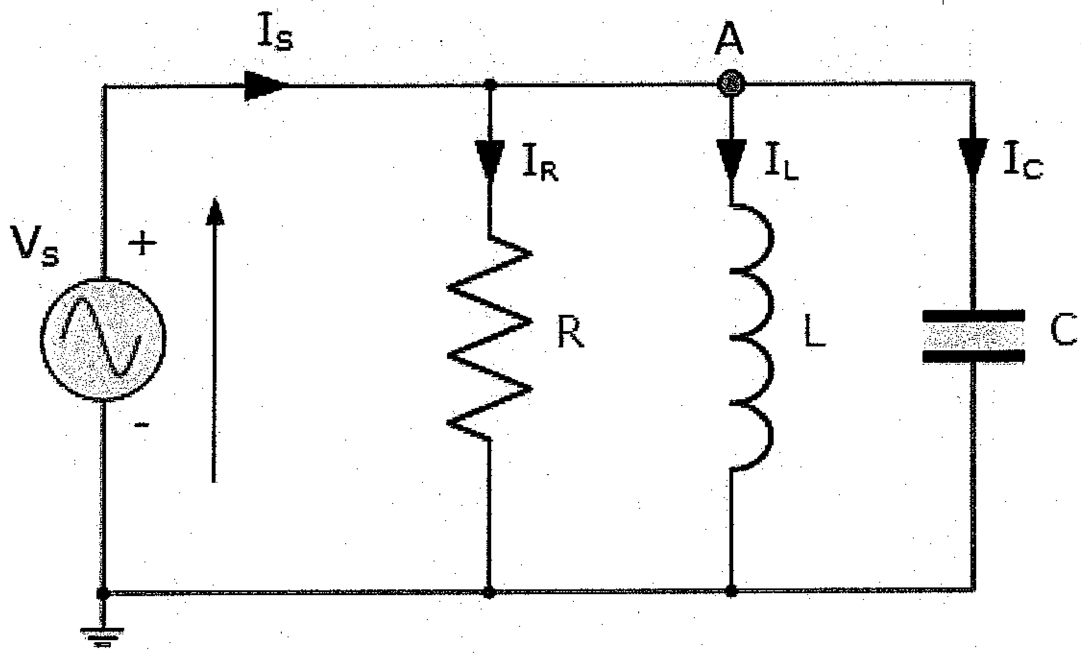
c) Write an equation for the electric potential at the center of the sphere, in terms of q and $\sigma(\theta,\phi)$, the surface charge density on the sphere. Take the potential at infinity to be zero.

d) What is the electric potential at the center of the sphere; take the potential at infinity to be zero.

e) Suppose now that the sphere is grounded. Find the total induced charge on the sphere, q'.



4) For the circuit shown below:



(a) Find the current, I_S as a function of V_S, the resistance R, the inductance L, and the capacitance C, and the angular frequency of the source, ω. Assume the time dependent voltage is given by

$$V_S e^{-i\omega t}.$$

(b) Does this circuit have a resonant frequency? If so, what is it in terms R, L and C? If not, why does it *not* have a resonant frequency?

5) Consider, quantum mechanically, a stream of particles of mass m, in one dimension; each is moving in the positive x direction with kinetic energy E toward a potential jump located at $x = 0$. The potential is zero for $x < 0$ and $3E/4$ for $x > 0$. What fraction of the particles are reflected at $x = 0$?

(6) The Hamiltonian of a one-dimensional quantum harmonic oscillator can be written as $H = \frac{\hbar\omega}{2}(-\partial_x^2 + x^2)$, where ω is the frequency of the oscillator, and x is a dimensionless spatial coordinate.

Define $a = \frac{(x+\partial_x)}{\sqrt{2}}$, $a^+ = \frac{(x-\partial_x)}{\sqrt{2}}$; the energy eigenstates are denoted by $|n\rangle = \frac{a^{+n}}{\sqrt{n!}}|0\rangle$, where $n = 0, 1, 2, \ldots$, and $|0\rangle$ is the ground state. Now an external field, $H' = x^{100}$, is applied to this harmonic oscillator.

(a) Calculate $|\langle 99|H'|0\rangle|^2$.

(b) Calculate $|\langle 100|H'|0\rangle\rangle|^2$.

(c) Calculate $|\langle 102|H'|0\rangle|^2$.

7) The Sun has radius R, and generates energy via fusion in the core with a high temperature, T_C. This fusion energy gets released in photons, but the photons will scatter many times before escaping from the Sun. Assume the Sun is a perfect black body at every radius, and that it is in a steady-state condition.

a) Determine the temperature of the outer region of the Sun (at radius R) if the core has a radius of βR, with $\beta < 1$.

b) Now determine the temperature of the Earth in terms of the distance between the Sun and the Earth, D, and the radius of the earth, R_E, using the result from part a). Assume that the Earth receives all of its energy from the Sun, that the Earth radiates as a perfect black body, and that the temperature is the same over the entire surface of the Earth.

8) Consider a simple model of a solid (held at fixed volume) as a set of N independent harmonic oscillators, each of frequency ω_1, and ground state energy $E_1 = N\varepsilon_1$ (including the zero point energy of the oscillators). Thus the energy of the solid in a particular state is given by

$$E = E_1 + \sum_{j=1}^{N} n_j \hbar\omega_1$$

where each n_j is an integer ranging from zero to infinity and is the number of excitations of oscillator j..

The solid can be in another phase with ground state energy $E_2 > E_1$ and $\omega_2 < \omega_1$ representing a less tightly bound crystal; here, $E_2 = N\varepsilon_2$

a) Compute the partition function and free energy of both phases.

b) What is the preferred phase at low temperature and what is the preferred phase at high temperature? Show that there is a phase transition.

c) Assuming that the phase transition happens in the regime of large temperature, determine that temperature of the transition.

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