Physics Graduate School Qualifying Examination

Fall 2018 Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet.** **Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
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- Use correct vector notation when appropriate.
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1) A motorcycle at speed v rides on a circular track at radius R. The track is tilted an angle θ above the horizontal. There is a coefficient of friction μ that resists the motorcycle from sliding down or up the track.
(a) Determine the speed v at which the motorcycle can circle the track even if $\mu = 0$.
(b) Is there a minimum speed, at which the motorcycle does not slide down the track, for the case of non-zero friction? If your answer is yes, give the value of this minimum speed.
(c) Is there a maximum speed, at which the motorcycle does not slide up the track, for the case of non-zero friction? If your answer is yes, give the value of this maximum speed.

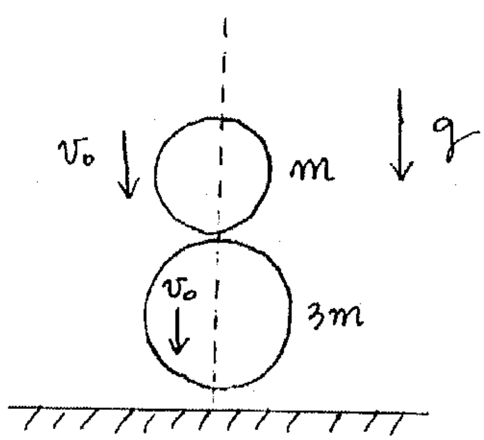
Your answers will depend on g, R, θ, and μ.

2. Consider two spheres whose masses are m and 3m (see figure). Both are initially held at height h, then allowed to fall vertically under gravity with the light sphere a small distance above the heavy sphere as shown. <u>The sizes of the spheres are negligible compared to the height h</u>. All collisions are elastic, and there is no air resistance.

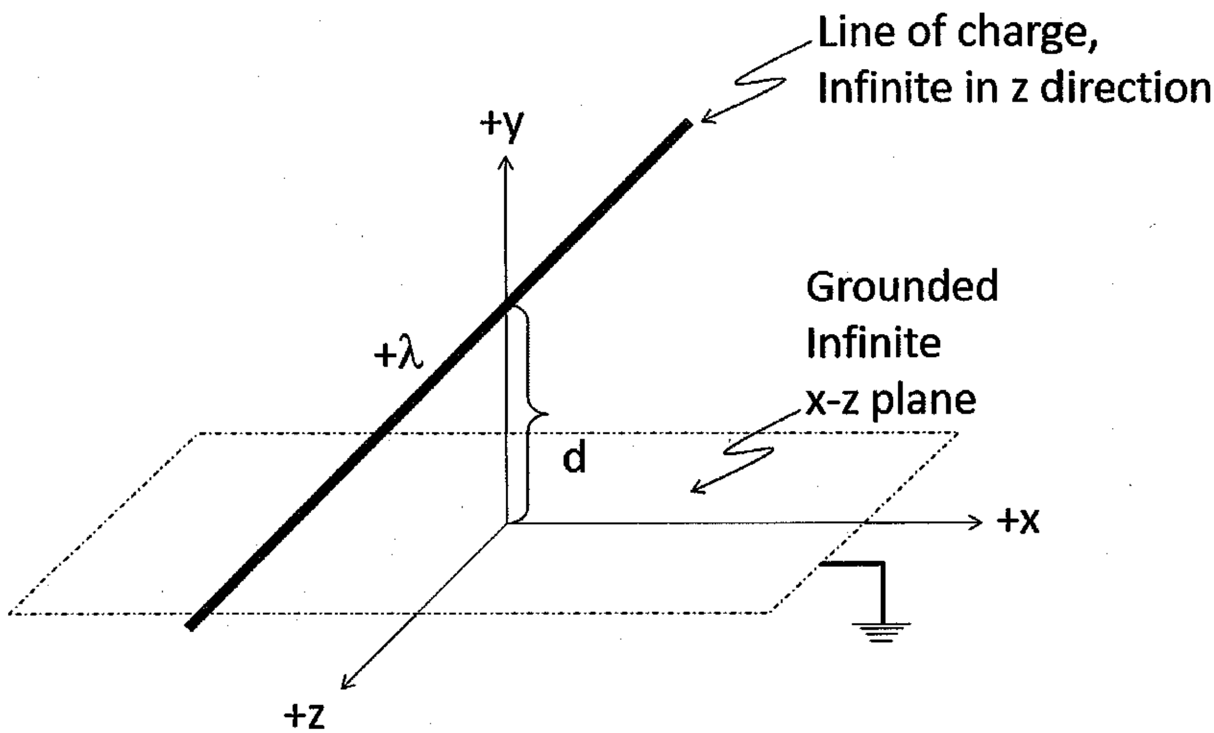
(a) Express the speed of the heavy sphere just before it hits the floor in terms of the quantities given.

(b) Compute the speed of the light sphere just after it collides with the heavy sphere. (The figure shows the two spheres just before the heavy sphere collides hits the floor.)

(c) Compute the height the light sphere reaches.

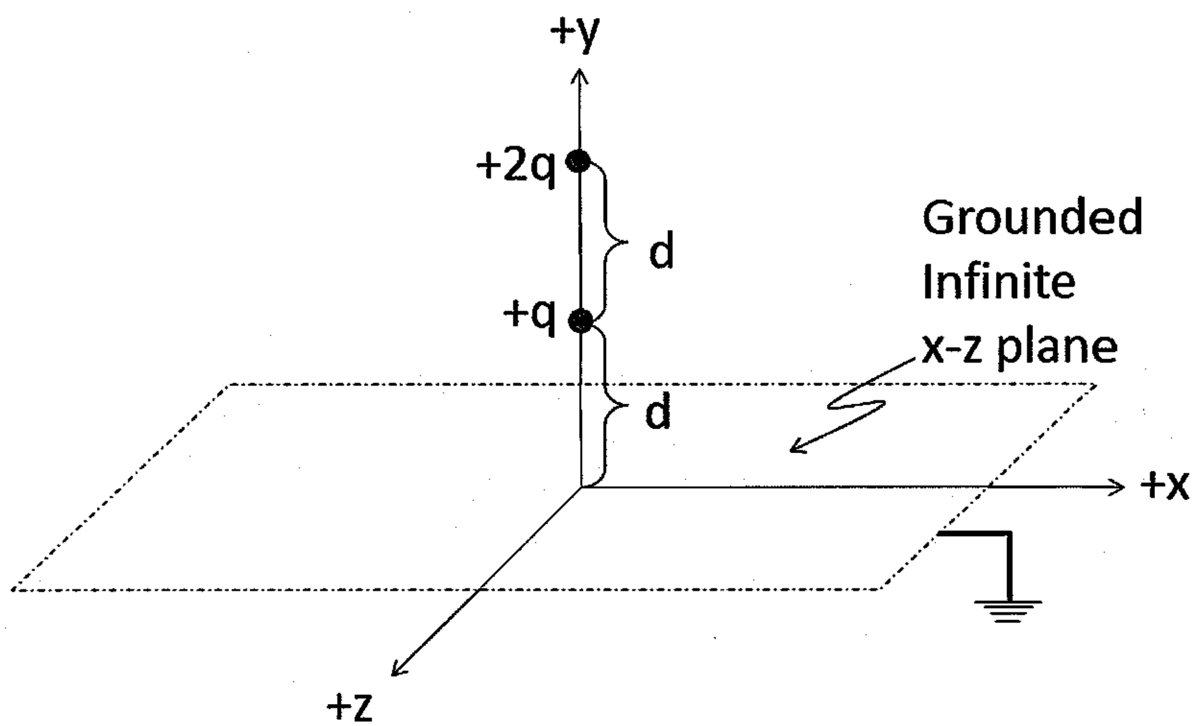


3) An infinite, grounded, conducting plane is in the xz-plane; its electric potential is thus held at V=0. A line of charge, with linear uniform charge density λ runs parallel to the z-axis, a distance d above the conducting plane. Assume the whole region exclusive of the conductors is vacuum.



(a) Compute the electric field $\vec{E}$, as a function of position, for y>0.
(b) Compute the electric potential, V, as a function of position, for y>0.
(c) Compute the capacitance per unit length of a thin wire of radius a, placed a distance d above a grounded plane. Assume that the wire radius is much smaller than d (i.e. $a \ll d$) so that the solution of part (a) is approximately correct in the region exclusive of the conductors.

4) The x-z plane is a **grounded conductor**; its electric potential is thus held at V=0. Two charges are placed as shown in the figure, +q at y=d, x=z=0, and +2q at y=2d, x=z=0.



(a) What is force on the $+q$ charge?

(b) What is the electric potential, $V(x, y, z)$ everywhere.

5) A quantum mechanical particle of mass M moves in one dimension in a harmonic oscillator potential, $V(x) = \frac{1}{2} M\omega^2 x^2$. The particle is in the ground state.

Then, at time t = 0, V(x) is abruptly changed, by changing $\omega \rightarrow 2\omega$.

a) What is the probability of finding the particle in the ground state of the new potential?

b) What is the probability of finding the particle in the first excited state of the new potential?

6) A two-state quantum system has the orthonormal basis |a> and |b>. Using a matrix notation in which the quantum state $|\psi\rangle = c_a |a\rangle + c_b |b\rangle$ is expressed as:

$$|\psi\rangle = \begin{pmatrix} c_a \\ c_b \end{pmatrix},$$ the Hamiltonian has the following form

$$H = \begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}$$

where Δ, Ω are real and positive.

a) What are the energies of the two eigenstates of H?

b) At time t=0, the system's wavefunction is given by |b>. What is the minimum time for the system to be found again in the state |b> with 100% probability?

7) Consider an ideal gas, satisfying, $p = NkT / V = nkT$ and $U = c_v NT$, where N is the number of atoms and c_v is a constant.

a) Derive an expression for $(\partial T / \partial p)_S$; this tells us how temperature changes with pressure during an adiabatic (constant entropy) process.

b) Next, we treat the atmosphere as an ideal gas, with altitude-dependent temperature, pressure, and density. In a simple model, the reason T(z) decreases with altitude z is that a rising air packet undergoes an adiabatic expansion, since p(z) is decreasing with altitude. Assume the equation of hydrostatic equilibrium,

$\frac{dp}{dz} = -g\rho(z)$ where $\rho = nM$ is the mass density, and M is the mass of an atom.

Compute $\frac{dT}{dz}$; express your answer in terms of g, M, k, c_v, and show that it is independent of altitude z.

8) Consider a gas of N classical particles in a three dimensional volume V, in the ultra-relativistic regime, namely such that, for a single particle

$$\varepsilon = pc$$

where ε is the energy, p the magnitude of the momentum, and c the speed of light. Assuming that Boltzmann statistics adequately describes this gas:

a) Compute the partition function and obtain the free energy as a function of (T,V,N)

b) Compute the entropy S, pressure P and average energy U of the gas as a function of (T,V,N).

c) Compute the specific heat at constant volume C_V

Useful Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

$$F = -kT \ln Z \qquad Z = \sum_i e^{-\frac{E_i}{kT}} \qquad dF = -SdT - pdV + \mu dN$$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \qquad -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi$$

Harmonic oscillator energy eigenfunctions: $\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$

$$\psi_n = \frac{(a_+)^n}{\sqrt{n!}} \psi_0 \qquad a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$$

$$\int_0^\infty dx \, x^N e^{-x} = N! \qquad \int_{-\infty}^{\infty} dx \, e^{-x^2} = \sqrt{\pi}$$

Physics Graduate School Qualifying Examination

Fall 2018 Part II

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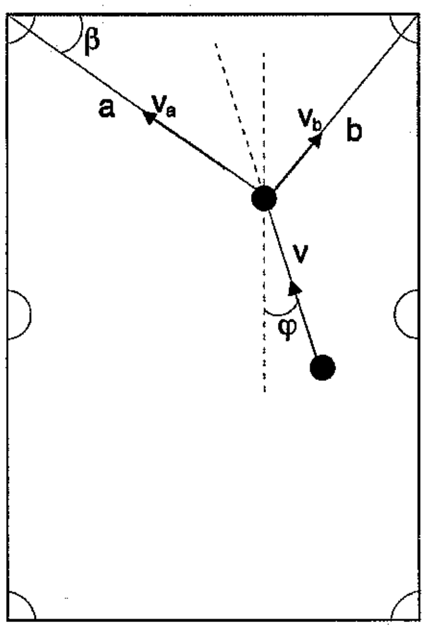
1) A small disc (target) rests at a point on the table, such that the connecting lines to the two corners are perpendicular to each other. An identical disc can be positioned at an arbitrary point on the table. The radii of the discs are negligible, you can treat them as point-like. Assume no friction, and that collisions are elastic.

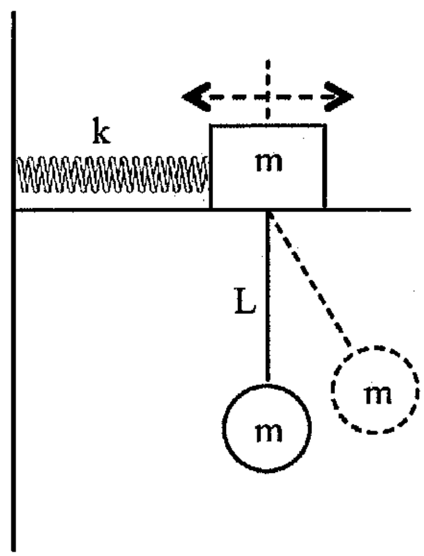


Determine the angle ϕ, shown in the figure, with which the second disc has to hit the target disc, to ensure that the two discs end up at the same time in the two corners.

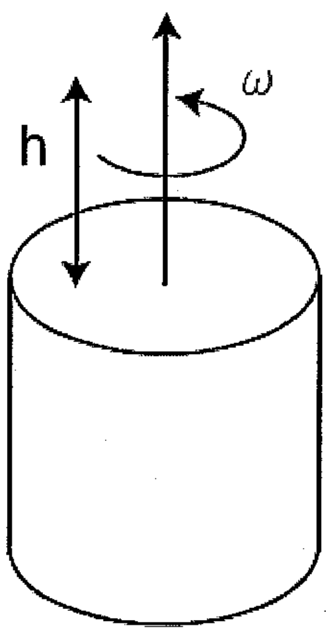
2. A pendulum of length L and mass m is connected to a block also of mass m that is free to move horizontally on a frictionless surface. The block is connected to a wall with a spring of spring constant k. For the special case where $k / m = g / L = \omega_0^2$ determine:

(a) The frequencies of the normal modes of this system for small oscillations around the stationary equilibrium configuration.

(b) If A and B are the amplitudes of the oscillations for the angle of the pendulum, and displacement of the block, respectively, determine how the two amplitudes are related.



3) Consider a cylinder of radius R and length L that is uniformly charged with charge density ρ. It occupies the volume $0 < x^2+y^2 < R^2$, $-L< z <0$. The cylinder rotates with a uniform angular velocity ω around the z-axis, which is also the center axis of the cylinder, as shown in the figure below.



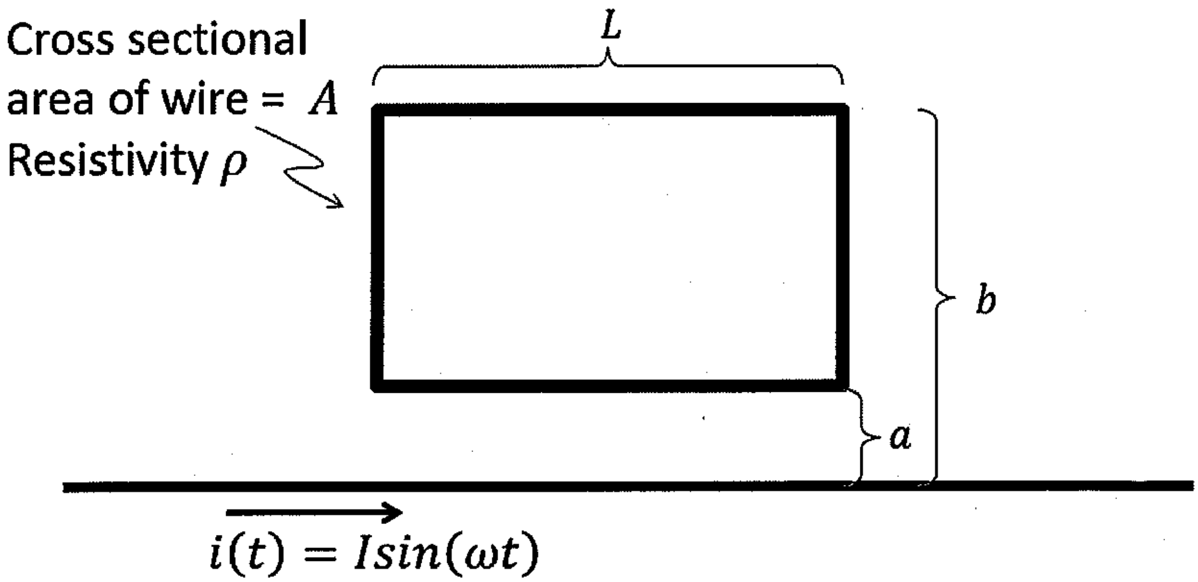
(a) Compute the electric current density, $\vec{J}$ as a function of distance, r, from the center of the cylinder. Here $r^2 = x^2 + y^2$

(b) Compute the magnetic induction, $\boldsymbol{B}$, along the z-axis, at x=y=0, z=h. Assume that h is very large $h \gg L,\ h \gg R$, and compute only the leading term for large h.

(c) If the total charge on the cylinder is kept the same, but redistributed such that the charge density obeys $\rho(r) = \beta r^n$ (n > 0) do you expect that the resulting magnetic field is smaller or larger than that you computed in part (b)? Explain!

Again, $r^2 = x^2 + y^2$.

4) The loop shown is near a wire with a time varying current $i(t) = I\sin(\omega t)$. The cross sectional area of the wire of the loop is A and its resistivity is ρ . Find an expression for the induced current $i_{ind}(t)$ in the loop.



5) A quantum mechanical particle in one dimension has a wave function given by

$$\psi(x) = c(1 - x^2 / L^2) \text{ for } |x| < L,$$

$$\psi(x) = 0 \text{ for } |x| > L$$

a) Compute the value of the normalizing constant c.

b) Compute $<p>$, $<p^2>$.

c) Compute $<x>$, $<x^2>$.

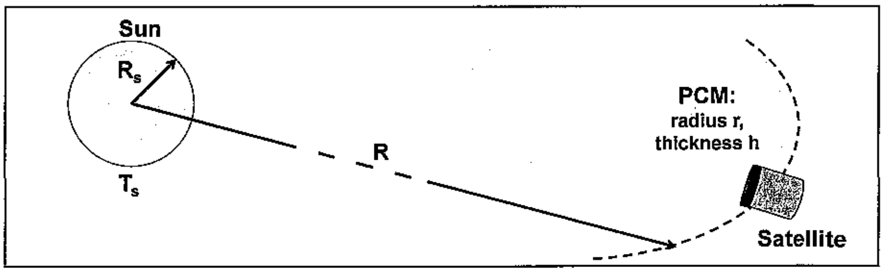
6) Consider a quantum mechanical harmonic oscillator in its ground state wavefunction.

a) Calculate the expectation value of the following operator: $\hat{A} = \hat{x}\hat{p}$.

b) Is it possible to measure the value of this operator? If your answer is "no", provide an explanation.

c) Consider the operator $\hat{B} = \hat{x}^2 \hat{p}^2$. Is it possible to measure the value of this operator? Explain your answer.

7) A satellite at a distance R from the sun ($R>>R_s$) contains a phase change material (PCM) used to store thermal energy. When exposed to solar radiation at normal incidence, the PCM efficiently absorbs energy causing a slow, uniform temperature increase.



Suppose the PCM completely fills a thermally insulated disk-shaped chamber of radius r and thickness h and has some initial temperature $T_i < T_m$ at t=0 where T_m is the temperature at which the PCM undergoes a phase transition.

The chamber faces the sun at normal incidence, as shown in the figure.

Neglecting any heat losses due to convection, conduction or radiation

a) How long will it take before the PCM completely changes phase assuming all incident radiation is absorbed?

b) What is the change in entropy of the PCM after the phase change is complete?

Use the following symbols when solving this problem:

Property	
mass density of PCM	ρ
specific heat per unit mass of PCM	**c**
latent heat of PCM	**L**
radius of sun	R_s
surface temperature of sun	T_s

8) Suppose we have two bodies, one at temperature T_1, the other at temperature T_2, with $T_1 > T_2$. The goal is to use a heat engine to extract the maximum amount of work W from the bodies, leaving them at the same final temperature T_F, and causing no other changes in the universe. Assume each body has a heat capacity C, which is a constant independent of temperature.

a) Calculate the maximum work W.

b) Calculate the corresponding final temperature T_F.

Express your answers in terms of C, T_1, and T_2.

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