Midterm Exam:

Date: Wednesday, March 6th
Time: 8:00 – 10:00 pm
Room: PHYS 203
Material: French, chapters 1-8
Review

1. Simple harmonic motion (one degree of freedom)
   – mass/spring, pendulum, water in pipes, RLC circuits
   – damped harmonic motion

2. Forced harmonic oscillators
   – amplitude/phase of steady state oscillations
   – transient phenomena

3. Coupled harmonic oscillators
   – masses/springs, coupled pendula, RLC circuits
   – forced oscillations

4. Uniformly distributed discrete systems
   – masses on string fixed at both ends
   – lots of masses/springs
Review

5. Continuously distributed systems (standing waves)
   – string fixed at both ends
   – sound waves in pipes (open end/closed end)
   – transmission lines
   – Fourier analysis

6. Progressive waves in continuous systems
   – dispersion, phase velocity/group velocity
   – reflection/transmission coefficients

7. Waves in two and three dimensions
   – Laplacian operator
   – Rotationally symmetric solutions in 2d and 3d
Coupled Discrete Systems

- The general method of calculating eigenvalues will always work, but for simple systems you should be able to decouple the equations by a change of variables.

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Forced Oscillations

• We mainly considered the qualitative aspects
  – We did not analyze the behavior when damping forces were significant

• Main features:
  – Resonance occurs at each normal mode frequency
  – Phase difference is $\delta = \pi/2$ at resonance

• Example: $x_A$ driven by the force $F(\omega) = F_0 \cos \omega t$
  – Calculate force term applied to normal coordinates
    $F_1(\omega) = F_2(\omega) = F_0 \cos \omega t$
  – Reduced to two one-dimensional forced oscillators:
    $\ddot{q}_1 + (\omega_0)^2 q_1 = F_0/m \cos \omega t$
    $\ddot{q}_2 + (\omega')^2 q_2 = F_0/m \cos \omega t$
Uniformly Distributed Discrete Systems

Equations of motion for masses in the middle:

\[
\ddot{x}_i + 2(\omega_0)^2 x_i - (\omega_0)^2 (x_{i-1} + x_{i+1}) = 0
\]

\[
(\omega_0)^2 = k/m
\]

\[
\ddot{y}_n + 2(\omega_0)^2 y_n - (\omega_0)^2 (y_{n+1} + y_{n-1}) = 0
\]

\[
(\omega_0)^2 = T/ml
\]
Uniformly Distributed Discrete Masses

- Proposed solution:

\[ x_n(t) = A_n \cos \omega t \]

\[ \frac{A_{n-1} + A_{n+1}}{A_n} = \frac{-\omega^2 + 2(\omega_0)^2}{(\omega_0)^2} \]

- We solved this to determine \( A_n \) and \( \omega_k \):

\[ A_{n,k} = C \sin \left( \frac{n k \pi}{N + 1} \right) \]

\[ \omega_k = 2 \omega_0 \sin \left( \frac{k \pi}{2(N + 1)} \right) \]

- General solution:

\[ x_n(t) = \sum_{k=1}^{N} a_k \sin \left( \frac{n k \pi}{N + 1} \right) \cos(\omega_k t - \delta_k) \]
Vibrations of Continuous Systems

- Amplitude of mass \( n \) for normal mode \( k \):
  \[
  A_{n,k} = C \sin \left( \frac{n k \pi}{N + 1} \right)
  \]

- Frequency of normal mode \( k \):
  \[
  \omega_k = 2 \omega_0 \sin \left( \frac{k \pi}{2(N + 1)} \right)
  \]

- Solution for normal modes:
  \[
  x_n(t) = A_{n,k} \cos \omega_k t
  \]

- General solution:
  \[
  x_n(t) = \sum_{k=1}^{N} a_k \sin \left( \frac{n k \pi}{N + 1} \right) \cos(\omega_k t - \delta_k)
  \]
Masses on a String

First normal mode

Second normal mode
This is the exact same problem as the previous two examples.
Forced Coupled Oscillators

• Qualitative features are the same:
  – Motion can be decoupled into a set of $N$ independent oscillator equations (normal modes)
  – Amplitude of normal mode oscillations are large when driven with the frequency of the normal mode
  – Phase difference approaches $\pi/2$ at resonance

• You should be able to anticipate the qualitative behavior when coupled oscillators are driven by a periodic force.
Continuous Distributions

Limit as $N \to \infty$ and $m/\ell \to \mu$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Boundary conditions specified at $x = 0$ and $x = L$:

- Fixed ends: $y(0) = y(L) = 0$
- Maximal motion at ends: $\dot{y}(0) = \dot{y}(L) = 0$
- Mixed boundary conditions

Normal modes will be of the form

$$y_n(x, t) = A_n \sin(k_n x) \cos(\omega_n t - \delta_n)$$

or

$$y_n(x, t) = A_n \cos(k_n x) \cos(\omega_n t - \delta_n)$$
Properties of the Solutions

\[ y(L, t) \sim \sin k_n L = 0 \quad \Rightarrow \quad k_n L = n\pi \]

<table>
<thead>
<tr>
<th>mode</th>
<th>wavelength</th>
<th>frequency</th>
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<tbody>
<tr>
<td>first</td>
<td>2L</td>
<td>( \frac{v}{2L} )</td>
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<tr>
<td>second</td>
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<tr>
<td>third</td>
<td>( \frac{2L}{3} )</td>
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<tr>
<td>fourth</td>
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\[ \lambda_n = \frac{2L}{n} \]

\[ \omega_n = \frac{n\pi v}{L} \]

\[ f_n = \frac{nv}{2L} \]
Boundary Conditions

- Examples:
  - String fixed at both ends: \( y(0) = y(L) = 0 \)
  - Organ pipe open at one end: \( \dot{y}(0) = \dot{y}(L) = 0 \)
    - Driving end has maximal pressure amplitude
  - Organ pipe closed at one end: \( \ddot{y}(0) = 0, y(L) = 0 \)
  - Transmission line open at one end: \( i(L) = 0 \)
  - Transmission line shorted at one end: \( v(L) \propto \frac{di(L)}{dt} = 0 \)
Fourier Analysis

• Normal modes satisfying $y(0) = y(L) = 0$:
  $$y_n(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \delta_n)$$

• General solution:
  $$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \delta_n)$$

• Initial conditions:
  $$y(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\delta_n) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$
  $$\dot{y}(x, 0) = \sum_{n=1}^{\infty} A_n \omega_n \sin\left(\frac{n\pi x}{L}\right) \sin(\delta_n) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$
Fourier Analysis

• Fourier sine transform:

\[ u(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \]

\[ B_n = \frac{2}{L} \int_0^L u(x) \sin \left( \frac{n\pi x}{L} \right) dx \]

• Fourier cosine transform:

\[ v(x) = \sum_{n=1}^{\infty} B_n \cos \left( \frac{n\pi x}{L} \right) \]

\[ B_n = \frac{2}{L} \int_0^L v(x) \cos \left( \frac{n\pi x}{L} \right) dx \]
Fourier Analysis

\[ B_n = A_n \cos \delta_n \]
\[ C_n = A_n \omega_n \sin \delta_n \]

Solve for amplitudes:

\[ A_n = \sqrt{B_n^2 + \frac{C_n^2}{\omega_n^2}} \]

Solve for phase:

\[ \tan \delta_n = \frac{C_n}{B_n \omega_n} \]
Fourier Analysis

• **Suggestion:** don’t simply rely on these formulas – use your knowledge of the boundary conditions and initial conditions.

• Example:
  – If you are given $\dot{y}(x, 0) = 0$ and $y(0) = y(L) = 0$ then you know that solutions are of the form
    \[ y(x, t) = \sum A_n \sin \left( \frac{n\pi x}{L} \right) \cos \omega_n t \]
  – If you are given $\dot{y}(x, 0) = 0$ and $\dot{y}(0) = 0, y(L) = 0$ then solutions are of the form
    \[ y(x, t) = \sum_{\text{odd } n} A_n \cos \left( \frac{n\pi x}{L} \right) \cos \omega_n t \]
Progressive Waves

• Far from the boundaries, other descriptions are more transparent:
  \[ y(x, t) = f(x \pm vt) \]

• The Fourier transform gives the frequency components:
  \[
  A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cos(kx) \, dx \\
  B(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \sin(kx) \, dx
  \]

\[
  g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \cos(kx) \, dk + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(k) \sin(kx) \, dk
  \]

• Narrow pulse in space \(\Rightarrow\) wide range of frequencies
• Pulse spread out in space \(\Rightarrow\) narrow range of frequencies
Properties of Progressive Waves

• Power carried by a wave:
  – String with tension $T$ and mass per unit length $\mu$
    $$P = \frac{1}{2} \mu \omega^2 A^2 \nu = \frac{1}{2} Z \omega^2 A^2$$

• Impedance of the medium:
  $$Z = \mu \nu = T / \nu$$

• Important properties:
  – Impedance is a property of the medium, not the wave
  – Energy and power are proportional to the square of the amplitude
Reflections

• Wave energy is reflected by discontinuities in the impedance of a system

• Reflection and transmission coefficients:
  – The wave is incident and reflected in medium 1
  – The wave is transmitted into medium 2
    \[ \rho = \frac{Z_1 - Z_2}{Z_1 + Z_2} \]
    \[ \tau = \frac{2Z_1}{Z_1 + Z_2} \]

• Wave amplitudes:
  \[ A_r = \rho A_i \]
  \[ A_t = \tau A_i \]
Reflected and Transmitted Power

• Power is proportional to the square of the amplitude.
  – Reflected power: \( P_r = \rho^2 P_i \)
  – Transmitted power: \( P_t = \tau^2 P_i \)

• You should be able to demonstrate that energy is conserved:
  
  \[
  \text{ie, show that } P_i = P_r + P_t
  \]
Dispersion

• Wave speed is sometimes a function of frequency.

• Phase velocity: \( v = \lambda f = \frac{\omega}{k} \) (constant)

• Group velocity: \( v_g = \frac{d\omega}{dk} \) (function of frequency)

• Energy that is carried by a pulse propagates with the group velocity

• In optics, \( v = c/n(k) \) and

\[
    v_g = v \left( 1 - \frac{k \, dn}{n \, dk} \right)
\]

(evaluated at the average wavenumber of the pulse)
Waves in Two and Three Dimensions

• Wave equation:

\[ \nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]

• When the function only depends on the radius, (eg, \( \frac{\partial \psi}{\partial \theta} = 0 \)) then this can be written:

\[ \nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]

Polar coordinates (2d)

\[ \nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \psi) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]

Spherical coordinates (3d)
Waves in Two Dimensions

• Wave equation in polar coordinates:

\[ \nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]

• Bessel’s equation:

\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\omega^2}{v^2} \psi = 0 \]

Let \( z = kr \) where \( k = \omega / v \)

\[ \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{z} \frac{\partial \psi}{\partial z} + \psi(z) = 0 \]

• Solutions: \( J_0(z) \sim \sqrt{\frac{2}{\pi}} \cos\left(z - \frac{\pi}{4}\right) \) and \( Y_0(z) \sim \sqrt{\frac{2}{\pi}} \sin\left(z - \frac{\pi}{4}\right) \)
Waves in Three Dimensions

- Wave equation in spherical coordinates:
  \[ \nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \psi) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]

- When \( \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi \) this is
  \[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \psi) + \frac{\omega^2}{v^2} \psi = 0 \]

- Solutions are of the form:
  \[ \psi(r, t) = A \frac{e^{ikr}}{r} \cos \omega t \]
Boundary Conditions in Two and Three Dimensions

• When a boundary condition imposes the restriction that $\psi(R, t) = 0$ then the function must have a node at $r = R$.

• Analogous to the 1-dimensional case:

This imposes the requirement that $kR$ is a root of the equation $f(kR) = 0$ which implies that $k_n = \frac{\omega_n}{v} = z_n/R$ where $z_n$ are roots of $f(z) = 0$. 
That’s all for now...

• Study these topics – make sure you understand the examples and assignment questions.

• Send e-mail if you would like specific examples discussed before the exam next Wednesday.

• Next topics: *waves applied to optics.*