Physics Graduate School Qualifying Examination

6 January 2000

Part I

Instructions: Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of yellow paper and use only one side of each sheet.** **Write your 3-digit student number on the upper right hand corner of each and every answer sheet, even if blank. All sheets which you receive should be handed in, even if blank.** All problems carry the same weight.

I-1. A particle of mass M moves in an external potential $V(x, y, z)$. Depending on the form of $V(x, y, z)$, the various components of the linear and the angular momentum may or may not be conserved.

For each of the following potentials, indicate whether or not the indicated components are conserved. (Write **Yes** or **No**.)

a) $V(x,y,z) = a\left(x^2 + y^2 + z^2\right)^3$

L_x - $\qquad P_x$ -
L_y - $\qquad P_y$ -
L_z - $\qquad P_z$ -

b) $V(x,y,z) = Ae^{a\left(y^2+z^2\right)}$

L_x - $\qquad P_x$ -
L_y - $\qquad P_y$ -
L_z - $\qquad P_z$ -

c) $V(x,y,z) = az^3$

L_x - $\qquad P_x$ -
L_y - $\qquad P_y$ -
L_z - $\qquad P_z$ -

d) $V(x,y,z) = az + b\left(x^2 + y^2\right)$

L_x - $\qquad P_x$ -
L_y - $\qquad P_y$ -
L_z - $\qquad P_z$ -

I-2. Suppose a current flows in the z direction, down an infinitely long cylindrical wire of radius R, with the following distribution (given in terms of cylindrical coordinates (r, ϕ, z)):

$$\vec{J} = ar^2\,\hat{z} \quad \text{for } r<R,$$
$$\vec{J} = 0 \quad \text{for } r>R.$$

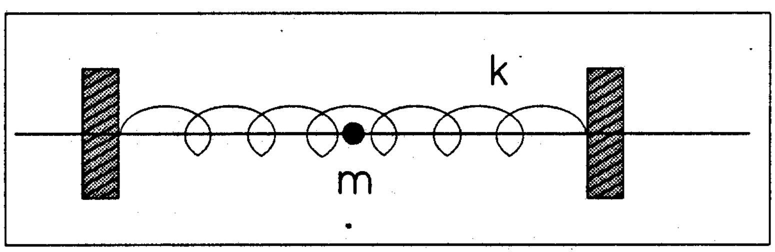
a) Compute the magnetic field for both $r<R$ and for $r>R$.
b) Compute the total current I flowing down the wire.

I-3. The plates of a parallel plate capacitor are distance d apart. One edge of the capacitor is held in contact with a liquid of dielectric constant k which has a mass density ρ. If a voltage V is applied by a battery, what is the equilibrium height of the liquid in the capacitor?

I-4. A three dimensional harmonic oscillator has a potential energy given by $V(r) = \frac{1}{2}Kr^2$. A particle of mass m is held by the potential. If the oscillator is in a state with quantum number N, what is the rms (root mean square) value for r (in other words, $\langle r^2 \rangle$ must be determined)?

I-5. Consider a particle of positive charge q and mass m that passes through the origin moving with speed v in the x direction at $t=0$. A uniform magnetic field $\boldsymbol{B}$ is parallel to the z direction. Find the particle's trajectory.

I-6. A quantum particle of mass m is confined in a 1-dimensional box where the two nearly hard walls are connected by a spring of force constant k and unperturbed equilibrium length, L_O. If the particle is in the third energy level (2nd excited state above ground state) what is the new equilibrium length, L, of the spring (assuming the particle frequency to be much higher than the natural frequency of the spring/walls system and the particle motion to be a small perturbation).



I-7. The half-life of a neutron is about 12 minutes, and the rest energy of a neutron is about 940 MeV. Estimate the kinetic energy required to give a neutron 12.5% probability of surviving a round trip from the earth to a star 25 light years away.

I-8. Consider the motion of a planet of mass m in the gravitational field of a dust clouded sun. Neglect the motion of the sun and dust cloud, imagine they are fixed in position and only the planet moves under the influence of the gravitational field $g(r)$ where

$$\vec{g}(\vec{r}) = \begin{cases} -G\left[\dfrac{\left(M_S - \frac{4\pi}{3}R_S^3\rho\right)}{|\vec{r}|^2} + \dfrac{4\pi}{3}\rho|\vec{r}|\right]\dfrac{\vec{r}}{|\vec{r}|} & , \quad R_S \le |\vec{r}| \le R \\ -G\left[\dfrac{\left[M_S + \frac{4\pi}{3}\left(R^3 - R_S^3\right)\rho\right]}{|\vec{r}|^2}\right]\dfrac{\vec{r}}{|\vec{r}|} & , \quad |\vec{r}| \ge R. \end{cases}$$

Here the sun is considered a sphere of radius R_S with a uniformly distributed total mass M_S, and is surrounded by a spherical cloud of dust of uniform density ρ extending from the sun's surface to a distance R from the sun's center.

a) What is the angular velocity, ω_0, of revolution of the planet in a circular orbit of radius r_0 (use the r_0 defining equation)?

b) Consider small radial displacements $\eta(t)$ about this orbit, $r(t) = r_0 + \eta(t)$, with $\eta << r_0$; find the angular frequency ω_r of these oscillations. Find and solve the equation of motion for $\eta(t)$.

For all of the above consider only the case where the orbit of the planet lies entirely within the dust cloud ($R_S < |\mathbf{r}| < R$). Use the simplified notation where $\omega^2 = 4\pi G\rho/3$ and $k = GM_S - 4\pi GR_S^3\rho/3 = GM_S - \omega^2 R_S^3$, so that the gravitational field within the dust cloud is expressed as

$$\vec{g}(\vec{r}) = -\left[\frac{k}{|\vec{r}|^2} + \omega^2|\vec{r}|\right]\frac{\vec{r}}{|\vec{r}|} \qquad , R_S \le |\vec{r}| \le R.$$

Also assume the gravitational force arising from the dust cloud is very much smaller than the sun's gravitational force on the planet, that is, $k/R^2 >> \omega^2 R$.

I-8. Continued

Hint: Set up Newton's 2nd law for the planet, $m\,\ddot{\vec{r}} = m\,\vec{g}(\vec{r})$, in cylindrical coordinates. Hence the position vector is given by $\vec{r} = r\,\hat{e}_r + z\,\hat{k}$, where here $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. Likewise, the basis unit vectors are given by

$$\hat{e}_r = \frac{x\,\hat{i} + y\,\hat{j}}{\sqrt{(x^2+y^2)}} \quad , \quad \hat{e}_\theta = \frac{-y\,\hat{i} + x\,\hat{j}}{\sqrt{(x^2+y^2)}} \quad , \quad \hat{e}_z = \hat{k} \ .$$

Note that in these coordinates $|\vec{r}| = \sqrt{(r^2 + z^2)}$, while $\dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{k}$ and

$$\ddot{\vec{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{e}_r + \left[r\ddot{\theta} + 2\dot{r}\dot{\theta}\right]\hat{e}_\theta + \ddot{z}\,\hat{k} \ .$$

A useful integral is

$$\int \frac{d\xi}{\sqrt{a^2 - \xi^2}} = \sin^{-1}\left(\frac{\xi}{a}\right)$$

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7 January 2000

Part II

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II-1. Due to the presence everywhere of the microwave radiation background, the minimum temperature possible of a gas in interstellar or intergalactic space is not 0 K but 2.7 K. This implies that a significant fraction of the molecules in space that possess excited states of low excitation energy may be in those excited states. Subsequent de-excitation leads to the emission of radiation that could be detected. Consider a (hypothetical) molecule with just one excited state.

a) What would the excitation energy have to be in order that a fraction f of the molecules be in the excited state?

b) Find the wavelength of the photon emitted in a transition to the ground state.

II-2. Two perfectly conducting thin spherical shells with radii a_1 and a_2 ($a_2 > a_1$) are placed concentrically. The space between the two shells is filled with a homogeneous substance of resistivity ρ. Calculate the resistance between the shells.

II-3. A thin uniform disc of radius R and mass M spins about an axis passing through its center; the axis makes an angle α with the normal to the disk. What torque is required to maintain a constant angular velocity ω? (Moments of inertia of the disk are: $MR^2/2$ about the normal passing through the center, $MR^2/4$ about a diameter.)

II-4. A soap bubble is held together by surface tension s, which is the surface energy per unit area. The bubble is prevented from collapse by pressure exerted by the trapped air. The air is kept at a constant temperature and obeys $pV = constant$, where p is the pressure, and V is the volume.

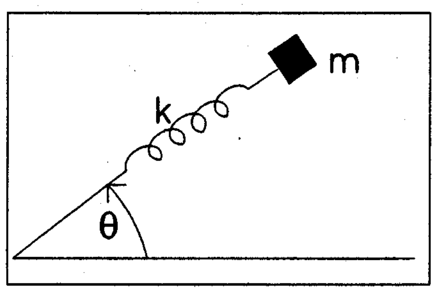
a) If the radius of the bubble is R, and the atmospheric pressure is p_A, what is the pressure inside the bubble?

b) A small electric charge Q is uniformly distributed on the surface of the bubble. Find the resulting change in the radius (to the first order in Q^2).

II-5. A mass m moves without friction on a horizontal plane and is connected to a fixed point by a spring of force constant k.

a) Write the Lagrangian for the system in terms of r and θ. Use the Lagrangian to obtain the equations of motion.

b) If the mass moves in a uniform circle of radius R with angular frequency ω, find an expression for R in terms of m, k, and r_0, the value of r when the spring is unstretched.

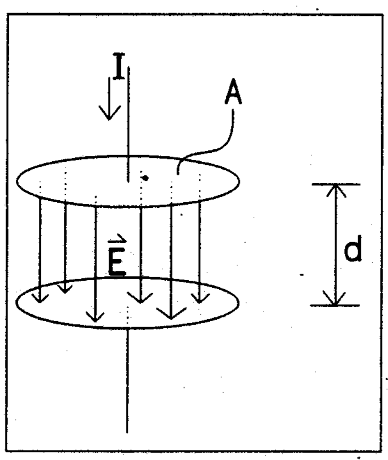


II-6. A circular parallel plate capacitor with area A and with distance d separating the plates is being charged. See the diagram below.

a) Determine in which direction the Poynting vector points.

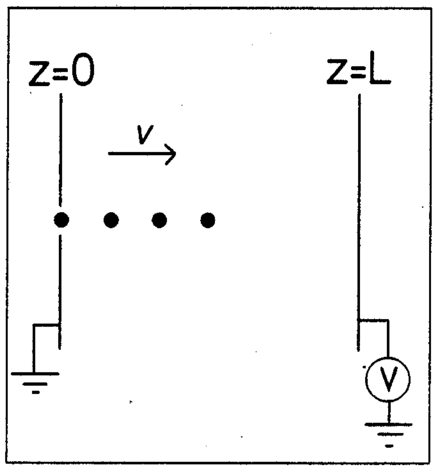
b) Calculate the rate at which energy flows into the cylindrical volume and verify that it is equal to the rate at which the stored electrostatic energy increases, that is

$$\int_{\substack{\text{Cylindrical}\\ \text{Boundary}}} \vec{S}\cdot\hat{n}\, da = -(Ad)\frac{d}{dt}\left(\frac{1}{2}\varepsilon_0 E^2\right)$$



II-7. Find the minimum voltage that may be applied to an x-ray tube to obtain a Bragg reflection at 90^0 to the incident beam if the incident beam is parallel to a cubic axis (100) of *NaCl*. The lattice spacing of *NaCl* is a=2.81A. (hc/e=12,400 V-A). (Leave answer as a fraction.)

II-8. A small steady electron source emits one electron every time interval Δt into a parallel plate capacitor.



a) What is the velocity, v, of the electrons as a function of z?

b) What is the average separation of the electrons at $z=0$, and at $z=L$?

c) What is the average magnetic field $B(r, z)$?